THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 24 (Last Week)
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Application: Markowitz Portfolio Selection

Harry Markowitz obatined the Nobel Prize in 1990.
Before discussing this application, we may first recall some contents in the **Probability Theory**. In the folloding, we define the following notations:

- $X = (x_1, \dots, x_n)$ be a random vector
- $m := \mathbb{E}[X] = (\mathbb{E}[x_1], \dots, \mathbb{E}[x_n])$
- $\Sigma = \mathrm{Var}[X] = \begin{pmatrix} \mathrm{Var}[x_1] & \mathrm{Cov}[x_1, x_2] & \cdots & \mathrm{Cov}[x_1, x_n] \\ \vdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \mathrm{Cov}[x_n, x_1] & \cdots & \cdots & \mathrm{Var}[x_n] \end{pmatrix}$

Let $b \in \mathbb{R}^n$, then $b^T X = \langle b, X \rangle$ is a random variable. By linearity and multivariate, we have

$$\mathbb{E}[b^T X] = b^T \mathbb{E}[X] = b^T m, \quad \text{and} \quad \mathrm{Var}[b^T x] = b^T \Sigma b$$

Introduction

- **Stock Price:** $S_t^k,\ t = 0, 1,\ k = 1, \dots, n$, where S_0^k is known and S_1^k is R.V.
- **Return:** $R^k := S_1^k - S_0^k$ (could be positive and negative)
- **Releative Return:** $X_k := \dfrac{R^k}{S_0^k} = \dfrac{S_1^k - S_0^k}{S_0^k}$, return by investing per \$1.
- **Portfolio:** $w = (x_1, \dots, w_n) \in \mathbb{R}^n$ and $\sum_{k=1}^{n} w_k = 1$, where w_k denotes *Ratio of money invested in Stock* k (total amount is \$1 and one invests w_k is stock k.)
- **Return of the Portfolio:** $R := \sum_{k=1}^{n} w_k x_k$ (naturally followed from the above)

Portfolio Optimization

Now, we are interested in the following problems:

$$\max_{w \in \mathbb{R}^n} \mathbb{E}[R] \qquad \text{(Maximize our profit)}$$

and

$$\min_{w \in \mathbb{R}^n} \mathrm{Var}[R] \qquad \text{(Minimize our risk)}$$

However, it is hard to achieve both cases above, i.e. **maximizing our profit** and **minimizing our risk** at the same time.
So, we may relax our problems as follows:

$$\max_{w\in\mathbb{R}^n} \mathbb{E}[R] \qquad \text{subject to} \quad \operatorname{Var}[R] \leq \sigma_0^2 \tag{1}$$

and

$$\min_{w\in\mathbb{R}^n} \operatorname{Var}[R] \qquad \text{subject to} \qquad \mathbb{E}[R] \geq \mu \tag{2}$$

Remarks. The above two problems (1) and (2) are **dual** problems.

Now, we will focus on the problem 2, and denote as

$$\min_{w\in\mathbb{R}^n} w^T\Sigma w \qquad \text{subject to} \quad \begin{cases} w^T m \geq \mu \\ \mathbb{1}^T w = 1 \end{cases} \tag{P}$$

where $m \in \mathbb{R}^n, \Sigma \in \mathcal{S}^n,\ \mu \in \mathbb{R}$ are given and $\mathbb{1} = (1,\ldots,1)^T$.

Remarks. Note that Σ is the covariance matrix, so $\Sigma \succ 0$ is symmetric positive definite. In our setting of the above problem, we assume that $n \geq 2$ and $\Sigma \succ 0$.

Problem Solving

To solve the problem P, we will proceed as follows:

- First, the above problem P is a **covex optimization problem** since $w \mapsto w^T\Sigma w$ is convex. For our simplicity, we may replace $w^T\Sigma w$ by $\frac{1}{2}w^T\Sigma w$ for easier differentiating.
- Let $f(x) = \frac{1}{2}w^T\Sigma w$, $g(w) = \mu - m^T w$ and $h(w) = \mathbb{1}^T w - 1$. Since $f(x) = \frac{1}{2}w^T\Sigma w$ is coercive function, so there exists optimizer w^*.
  Then, the **Qualification condition holds** under Slater condition: there exists w such that

  $$g(w) < 0 \iff m^T w > \mu$$

  and $\nabla h(w) = \mathbb{1}^T \neq \mathbf{0}$ linearly independent.
- By the KKT theorem, there exists $\lambda \geq 0$ and $\gamma \in \mathbb{R}$ such that

  $$\begin{cases} \nabla f(w^*) + \lambda \nabla g(w^*) + \gamma \nabla h(w^*) = \mathbf{0} \\ \lambda g(w^*) = 0 \end{cases} \iff \begin{cases} \Sigma w^* - \lambda m + \gamma \mathbb{1} = \mathbf{0} \\ \lambda(m^T w^* - \mu) = 0 \end{cases} \tag{3}$$

  together with $\lambda \geq 0, m^T w^* \geq \mu$ and $\mathbb{1}^T w^* = 1$.
- Then, we divide into the following cases:
  - **Case 1:** $m^T w^* - \mu > 0$
    Then, from $\lambda(m^T w^* - \mu) = 0$ we can deduce that $\lambda = 0$.
    Putting back to the first equation of 3, we get $\Sigma w^* + \gamma \mathbb{1} = \mathbf{0}$, then

    $$w^* = (-\gamma)\Sigma^{-1}\mathbb{1} \qquad \text{for some } \gamma \in \mathbb{R}.$$

    From $\mathbb{1}^T w^* = 1$, we have $w^* = \dfrac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$.

Remarks. If we consider an alternative problem (P') without the inequality constraint as

$$\min_{w\in\mathbb{R}^n} w^T\Sigma w \qquad \text{subject to} \quad \mathbb{1}^T w = 1 \tag{P'}$$

Then by the KKT theorem, there exists $\gamma \in \mathbb{R}$ such that

$$\begin{cases} \Sigma w^* + \gamma\mathbb{1} = 0 \\ \mathbb{1}^T w^* = 1 \end{cases}$$

and then finally we can still solve $w^* = \dfrac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$.

Indeed, if we define $w^*_{mv} = \dfrac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$ and $m^T w^*_{mv} - \mu > 0$, then w^*_{mv} is the solution to the problem (P).

- **Case 2:** $m^T w^* = \mu$

  Then we have

$$\begin{cases} \mathbb{1}^T w^* = 1 \\ \Sigma w^* - \lambda m + \gamma\mathbb{1} = \mathbf{0} \end{cases}$$

Multiplying Σ^{-1} to the left on both sides of the second equation gives

$$w^* = \Sigma^{-1}\left(\Sigma w^*\right) = \lambda\Sigma^{-1}m - \gamma\Sigma^{-1}\mathbb{1}$$

Putting back to $\mathbb{1}^T w^* = 1$ and $m^T w^* = \mu$, we have

$$\begin{cases} m^T\left(\lambda\Sigma^{-1}m - \gamma\Sigma^{-1}\mathbb{1}\right) = \mu \\ \mathbb{1}^T\left(\lambda\Sigma^{-1}m - \gamma\Sigma^{-1}\mathbb{1}\right) = 1 \end{cases}$$

Rewrite as the matrix form, we have

$$\underbrace{\begin{pmatrix} m^T\Sigma^{-1}m & -m^T\Sigma^{-1}\mathbb{1} \\ \mathbb{1}^T\Sigma^{-1}m & -\mathbb{1}^T\Sigma^{-1}\mathbb{1} \end{pmatrix}}_{\text{symmetric matrix}} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \begin{pmatrix} \mu \\ 1 \end{pmatrix}$$

and

$$\delta := \det\begin{pmatrix} m^T\Sigma^{-1}m & -m^T\Sigma^{-1}\mathbb{1} \\ \mathbb{1}^T\Sigma^{-1}m & -\mathbb{1}^T\Sigma^{-1}\mathbb{1} \end{pmatrix} \begin{cases} = 0 & \text{if } m \neq c\mathbb{1} \text{ for some } c \in \mathbb{R} \\ > 0 & \text{if otherwise} \end{cases}$$

So, let us just consider the condition $m \neq c\mathbb{1}$ for all $c \in \mathbb{R}$ to ensure the matrix is invertible. Now, to solve the above equation, we obtain a solution $(\lambda, \gamma) = (\mathbb{1}^T v, -m^T v)$, where $v = \delta^{-1}\Sigma^{-1}(\mu\mathbb{1} - m)$.

Finally, we can find the optimal solution as

$$w^* = (1-\alpha)\frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}} + \alpha\frac{\Sigma^{-1}m}{\mathbb{1}^T\Sigma^{-1}m} = (1-\alpha)w^*_{mv} + \alpha w^*_{mk}$$

where $w^*_{mv} = \dfrac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$ and $w^*_{mk} = \dfrac{\Sigma^{-1}m}{\mathbb{1}^T\Sigma^{-1}m}$ are two special porfolios, and

$$\alpha = \frac{\mu(m^T\Sigma^{-1}\mathbb{1})(\mathbb{1}^T\Sigma^{-1}\mathbb{1}) - (m^T\Sigma^{-1}\mathbb{1})^2}{\delta}$$

Conclusion

- Step 1: Construct two portfolios: w_{mv}^* and w_{mk}^*.
- Step 2: If $m^T w_{mv}^* > \mu$, then w_{mv}^* is the optimal portfolio, the solution to (P).
- Step 3: If $m^T w_{mv}^* \leq \mu$, then $(1-\alpha)w_{mv}^* + \alpha w_{mk}^*$ is the solution.

— End of Lecture 24 —